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GENERAL ELEMENTS OF IDEALS IN LOCAL RINGS

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In many situations arising in the theory of local rings, it is necessary to make use of elements x_1, \dots, x_s of ideals $\mathcal{O}_1, \dots, \mathcal{O}_s$ which are sufficiently general in some sense, depending on the particular situation involved. The purpose of this lecture is to describe a general set-up in which such general elements can be defined which satisfy the required conditions in most such situations and to give an illustration of its application.

We suppose that (Q, \mathfrak{m}, k) is a local ring of dimension d . We first construct the general extension Q_g of Q . Let X_1, X_2, \dots be a countable sequence of indeterminates over Q . Then Q_g is the localisation of $Q[X_1, X_2, \dots]$ at the prime ideal $\mathfrak{m}[X_1, X_2, \dots]$. It follows from a general result of Grothendieck that Q_g is noetherian (alternatively one can prove that if \mathcal{O} is a finitely generated ideal of Q_g , then $\bigcap_{n=1}^{\infty} (\mathcal{O} + \mathfrak{m}_g^n) = \mathcal{O}$, and then, observing that the completion of Q_g is noetherian, use the above to show that if $\mathcal{O}\overline{Q}_g = \mathcal{O}'\overline{Q}_g$ where \mathcal{O}' is a finitely generated ideal of Q_g contained in \mathcal{O} , then $\mathcal{O} = \mathcal{O}'$.)

Now suppose that $\mathcal{O}_1, \dots, \mathcal{O}_s$ are ideals of Q , and that \mathcal{O}_1 has a basis a_{11}, \dots, a_{im_1} . Write $M_1 = m_1 + \dots + m_s$. Then we term x_1, \dots, x_s an independent set of general elements of $\mathcal{O}_1, \dots, \mathcal{O}_s$ if there exists an automorphism T of Q_g over Q such that

$$T(x_i) = \sum_{j=1}^{m_i} X_{M_{i-1}+j} a_{ij} \quad (i = 1, \dots, s).$$

It is a simple matter to prove that this definition is independent of the choice of bases of $\mathcal{O}_1, \dots, \mathcal{O}_s$. It also follows that the ideal $(x_1, \dots, x_s) \cap Q$ of Q and the Q -algebra $Q_g/(x_1, \dots, x_s)$ (to within isomorphism as a Q -algebra) depend only on the ideals $\mathcal{O}_1, \dots, \mathcal{O}_s$. I will only consider the first in the case when the ideals $\mathcal{O}_1, \dots, \mathcal{O}_s$ are all equal to \mathcal{O} . Let $a(\mathcal{O})$ denote the analytic spread of \mathcal{O} , and $v(\mathcal{O})$ the minimal number of generators of \mathcal{O} . Then

- i) if $s < a(\mathcal{O})$, the ideal $(x_1, \dots, x_s) \cap Q$ is nilpotent;
- ii) if $s = a(\mathcal{O})$, (x_1, \dots, x_s) is a reduction of $\mathcal{O}Q_g$ and hence $(x_1, \dots, x_s) \supseteq \mathcal{O}^n Q_g$ for n large, and hence $(x_1, \dots, x_s) \cap Q$ contains a power of \mathcal{O} ;
- iii) if $s \geq v(\mathcal{O})$, we have $(x_1, \dots, x_s) \cap Q = \mathcal{O}$.

Now we consider the second. In this case we will be concerned with the case when $s = d-1$ or d , and the ideals $\mathcal{O}_1, \dots, \mathcal{O}_s$ are all \mathcal{M} -primary. Let N be any integer and define Q_N to be the ring $Q[Y_1, \dots, Y_N]$ localised at $\mathcal{M}[Y_1, \dots, Y_N]$, Y_1, \dots, Y_N being indeterminates over Q . If we replace Y_i by X_i , it is clear that we can consider Q_N as a subring of Q_g . Now suppose that \mathcal{O} is any ideal of Q_g . Then for some N , \mathcal{O} is generated by elements of the sub-ring Q_N of Q_g and therefore $\mathcal{O} = (\mathcal{O} \cap Q_N)Q_g$. Now we have an isomorphism of $(Q_g)_N \rightarrow Q_g$ in which X_i maps to X_{N+i} and $Y_i \rightarrow X_i$ for $i = 1, \dots, N$. It follows that Q_g/\mathcal{O} is isomorphic to

$(Q_g)_N/\alpha^*$, where α^* is an ideal of $(Q_g)_N$ meeting Q_g in $(Q \cap \alpha)Q_g$. The case that will concern us is when α is generated by general elements x_1, \dots, x_{d-1} of \mathfrak{m} -primary ideals $\alpha_1, \dots, \alpha_{d-1}$ of Q . For simplicity of exposition, we will restrict ourselves to the case when Q is a domain. Then $Q_g/(x_1, \dots, x_{d-1})$ is a local ring of dimension 1. Now suppose y_i, z_i ($i = 1, \dots, d-1$) is a set of independent general elements of the ideals $\alpha_1, \alpha_1, \dots, \alpha_{d-1}, \alpha_{d-1}$. Now choose N so that the elements y_i, z_i ($i = 1, \dots, d-1$) are all contained in the sub-ring Q_N of Q_g . Then it is not difficult to prove that the elements $w_i = y_i - x_{N+i}z_i$ ($i = 1, \dots, d-1$) form a set of independent general elements of $\alpha_1, \dots, \alpha_{d-1}$. We further note that for each i , the elements y_i, z_1, \dots, z_{d-1} generate an $\mathfrak{m}Q_g$ -primary ideal of Q_g . We now quote a general result which will be proved in an appendix:

Let Q be a local domain of dimension d , and let y_i, z_i ($i = 1, \dots, d-1$) be elements of Q such that y_i, z_1, \dots, z_{d-1} generate an \mathfrak{m} -primary ideal for each i . Then, if B is the ring

$$Q[y_1/z_1, \dots, y_{d-1}/z_{d-1}],$$

i) $B/\mathfrak{m}B$ is isomorphic to $k[X_1, \dots, X_{d-1}]$, where $k = Q/\mathfrak{m}$, and X_1, \dots, X_{d-1} are indeterminates over k ;

ii) if L denotes B localised at the prime ideal $\mathfrak{m}[y_1/z_1; \dots, y_{d-1}/z_{d-1}]$, and $Q(X)$ denotes the ring $Q[X_1, \dots, X_{d-1}]$ localised at $\mathfrak{m}[X_1, \dots, X_{d-1}]$, where X_1, \dots, X_{d-1} are indeterminates over Q , then the kernel of the homomorphism of $Q(X)$ onto L in which $X_i \mapsto y_i/z_i$ ($i = 1, \dots, d-1$) is a prime ideal \mathfrak{P} containing the ideal $\mathfrak{X} = (y_1 - z_1 X_1, \dots, y_{d-1} - z_{d-1} X_{d-1})$ and $\mathfrak{P}/\mathfrak{X}$ is annihilated by a power of \mathfrak{m} .

Applying this result, we see that, replacing Q by Q_g and giving y_i, z_i their original meaning, the ring L obtained in this situation is isomorphic to $Q_g/(x_1, \dots, x_{d-1}) : \mathfrak{m}^n$ if n is large enough.

It follows that we can consider L in two ways, first as a homomorphic image of Q_g , and second as a local ring containing Q_g and contained in its field of fractions F_g . Further the maximal ideal of L is $\mathfrak{m}L$ and $\mathfrak{m}L \cap Q_g = \mathfrak{m}Q_g$. Now L is 1-dimensional. Hence, by the Krull-Akizuki theorem, the integral closure L^* of L in F_g is the intersection of a finite set of discrete valuation rings. Let the associated valuations be V_1, \dots, V_q and let their restriction to the field of fractions F of Q be v_1, \dots, v_q . Then v_1, \dots, v_q are independent of the choice of the elements y_i, z_i .

Now we must digress to consider valuations on Q_g . Suppose that V is a valuation ≥ 0 on Q_g , and > 0 on $\mathfrak{m}Q_g$, and taking integer values. If K_V is the residue field of V , then K_V is an extension of k_g , and an old result of Zariski states that $\text{tr.deg}_{k_g} K_V \leq d-1$. Now let v be the restriction of V to F . Then it is quite easy to prove that

$$\text{tr.deg}_{k_g} K_V \geq \text{tr.deg}_k K_v.$$

Now I recall another old result, due in this case to Northcott. Let K denote the residue field of L (which is a pure transcendental extension of k_g of transcendence degree $d-1$). Now the valuations V_i already referred to have an extension to

the completion \bar{L} of L which we denote by \bar{V}_i , and each such extension \bar{V}_i takes the value ∞ on a minimal prime ideal \mathfrak{P}_i of \bar{L} . Let δ_i denote the length of the primary component of (0) in \bar{L} with associated prime \mathfrak{P}_i . Then if $x \in L$,

$$e(xL) = \ell(L/xL) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(x)$$

where $e(\cdot)$ is the multiplicity.

Now we turn to multiplicities and degree functions. Following Teissier, we will use mixed multiplicities. Let $\alpha_1, \dots, \alpha_d$ be d \mathcal{M} -primary ideals of Q , and let M be a finitely generated Q -module. Then we define $e(\alpha_1, \dots, \alpha_d; M)$ as $e(x_1, \dots, x_d; M)$ where x_1, \dots, x_d are independent general elements of $\alpha_1, \dots, \alpha_d$. Then we have the result that if L is as described earlier,

$$e(\alpha_1, \dots, \alpha_d) = e(x_d L) = e(\alpha_d L),$$

the latter following since $x_d L$ is a reduction of $\alpha_d L$. Further this latter remark also implies that, if V_i, v_i have the meanings given earlier, then $V_i(x_d) = v_i(\alpha_d)$ where the latter denotes the minimum value of $v_i(x)$ on α_d . We further note that $e(\alpha_1, \dots, \alpha_d; M)$ is a symmetric function of $\alpha_1, \dots, \alpha_d$ and, if α'_d is another \mathcal{M} -primary ideal of Q , then

$$e(\alpha_1, \dots, \alpha_d, \alpha'_d; M) = e(\alpha_1, \dots, \alpha_d; M) + e(\alpha_1, \dots, \alpha'_d; M)$$

we can now write down a formula for the multiplicity symbol

$$e(\alpha_1, \dots, \alpha_d; Q) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(\alpha_d)$$

and similar formulae arising from the symmetry of the symbol. However this formula attains its full force if we introduce

degree functions. We define the degree function $d(\alpha_1, \dots, \alpha_{d-1}; x)$ where x is an element of Q to be $e(\alpha'_1, \dots, \alpha'_{d-1}; Q')$ where $Q' = Q/x$ and $\alpha'_i = (\alpha_i + xQ)/xQ$. If Q is a domain, this can also be written as $e(x_1, \dots, x_{d-1}, x; Q)$ and we obtain the expression

$$d(\alpha_1, \dots, \alpha_{d-1}; x) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(x).$$

APPENDIX

First we prove a lemma which is well known.

LEMMA. Let B be a noether domain, y, z elements of B such that (y, z) has height 2. Let B' be the ring $B[y/z]$ and let \mathfrak{P} be the kernel of the map $B[Y] \rightarrow B'$ in which $Y \mapsto y/z$. Then \mathfrak{P} contains $w = zY - y$, and

$$wB[Y] : (z^m, y^m) = \mathfrak{P}$$

if m is sufficiently large. Further, if \mathfrak{m} is any prime ideal of B containing (y, z) , then $B'/\mathfrak{m}B' \cong (B/\mathfrak{m})[X]$, where X is an indeterminate over $B'/\mathfrak{m}B'$.

Proof. Let $f(Y)$ be a polynomial of degree r over B such that $f(y/z) = 0$. Then we can write $f(Y) = F(Y, 1)$ where $F(Y, Z)$ is a homogeneous polynomial over B of degree r such that $F(y, z) = 0$. Then

$$\begin{aligned} z^r F(Y, Z) &= F(zY, zZ) = F(yZ + (zY - yZ), zZ) \\ &= F(yZ, zZ) + (zY - yZ)G(Y, Z) \quad \text{by Taylor's Theorem} \\ &= z^r F(y, z) + (zY - yZ)G(Y, Z) \end{aligned}$$

whence, by putting $Z = 1$, we see that $z^r f(Y) \in wB[Y]$. Also,

$$y^r f(Y) = (y^r - z^r Y^r) f(Y) + Y^r z^r f(Y) \in wB[Y].$$

But as the ascending sequence of ideals $wB[Y] : (y^r, z^r)$ becomes stationary for large r , it follows that

$$\mathfrak{P} = wB[Y] : (y^m, z^m) \quad m \text{ large.}$$

Hence \mathfrak{P} is the radical of $wB[Y]$ and since $y, z \in \mathcal{M}$, $w \in \mathcal{M}B[Y]$, i.e. $\mathfrak{P} \subset \mathcal{M}B[Y]$, which proves the result.

We now come to the main result of this appendix.

THEOREM. Let (Q, \mathcal{M}, k) be a local domain of dimension $d \geq 2$, and let y_i, z_i ($i = 1, \dots, d-1$) be elements of \mathcal{M} such that $(y_i, z_1, \dots, z_{d-1})$ is \mathcal{M} -primary for $i = 1, \dots, d-1$. Let $u_i = y_i/z_i$ and $B = Q[u_1, \dots, u_{d-1}]$. Then

$$B/\mathcal{M}B \cong k[X_1, \dots, X_{d-1}]$$

where X_1, \dots, X_{d-1} are indeterminates over k , implying that $\mathcal{M}B$ is prime.

Further let $L = B_{\mathcal{M}B}$ and let Q_{d-1} denote $Q[X_1, \dots, X_{d-1}]$ localised at $\mathcal{M}[X_1, \dots, X_{d-1}]$. Let \mathfrak{P} denote the kernel of the homomorphism $Q_{d-1} \rightarrow L$ in which $X_i \rightarrow u_i$. Let $w_i = z_i X_i - y_i$ and let \mathfrak{X} be the ideal (w_1, \dots, w_{d-1}) . Then for r large,

$$\mathcal{M}^r \mathfrak{P} \subset \mathfrak{X}.$$

Proof. The proof will be by induction on d , the case $d=2$ following from the lemma. Now suppose that $d > 2$. Write Q' for $Q[u_{d-1}]$ localised at $\mathcal{M}[u_{d-1}]$, which is prime by the lemma. We first prove that $(y_i, z_1, \dots, z_{d-2})Q'$ is $\mathcal{M}Q'$ -primary for $i = 1, \dots, d-2$. Now, by the lemma, $Q' \cong Q(X_{d-1})/\mathfrak{P}'$, where $Q(X_{d-1})$ denotes $Q[X_{d-1}]$ localised at $\mathcal{M}[X_{d-1}]$, and \mathfrak{P}' is the radical of $w_{d-1}Q(X_{d-1})$. Hence it will be sufficient to show that $(w_{d-1}, y_i, z_1, \dots, z_{d-2})$ is $\mathcal{M}Q(X_{d-1})$ -primary. Write

$$C_i = y_i Q(X_{d-1}) + z_1 Q(X_{d-1}) + \dots + z_{d-2} Q(X_{d-1}).$$

Then the minimal prime ideals of C_i are generated by elements of Q and so can only contain w_{d-1} if it contains y_{d-1}, z_{d-1} . Since $C_i + z_{d-1} Q(X_{d-1})$ is \mathfrak{m} -primary, $\dim C_i = 1$, and since w_{d-1} belongs to no minimal prime of C_i , the result now follows.

Now we consider the first statement of the theorem. It is clearly equivalent to the statement that if $f(X_1, \dots, X_{d-1})$ is a polynomial over Q such that $f(u_1, \dots, u_{d-1}) = 0$, then all the coefficients of f belong to \mathfrak{m} . Suppose there is a coefficient of f not in \mathfrak{m} . Then if we consider the polynomial $f(X_1, \dots, X_{d-2}, u_{d-1})$ as a polynomial with coefficients in Q' , then the lemma implies that this has a coefficient not in $\mathfrak{m}Q'$. But Q' has dimension $d-1$ and the conditions of the theorem apply. Hence by our inductive hypothesis $f(u_1, \dots, u_{d-1}) \neq 0$.

We are now in a position to construct L . Consider the homomorphism $Q_{d-1} \rightarrow L$. This can be factored as the product of the homomorphism $Q_{d-1} \rightarrow Q'_{d-2}$ in which $X_{d-1} \rightarrow u_{d-1}$ and the homomorphism $Q'_{d-2} \rightarrow L$. Denote by \mathcal{O} the kernel of the homomorphism $Q_{d-1} \rightarrow Q'_{d-2}$. Applying the inductive hypothesis to the second factor, we see that, for r large,

$$\mathfrak{m}^r \mathfrak{P} \subset \mathcal{O} + (w_1, \dots, w_{d-2})$$

while, by the lemma,

$$(y_{d-1}^m, z_{d-1}^m) \mathcal{O} \subset w_{d-1} Q_{d-1}.$$

Hence

$$(y_{d-1}^m, z_{d-1}^m) \mathfrak{m}^r \mathfrak{P} \subset (w_1, \dots, w_{d-1}) = \mathfrak{X}.$$

But by reordering the suffixes $1, \dots, d-1$, we can replace $d-1$ on the left hand side by i ($i = 1, \dots, d-2$). Hence if m, r are

large enough,

$$(y_1^m, \dots, y_{d-1}^m, z_1^m, \dots, z_{d-1}^m)^{m^r} \mathfrak{p} \subset \mathfrak{I}$$

and the result follows since the first factor is m -primary.